Maximal Order of an NG-group

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Abstract
This study was aimed to consider the NG-group that consisting of transformations on a nonempty set $A$ has no bijection as its element. In addition, it tried to find the maximal order of these groups. It found the order of NG-group not greater than $n$. Our results proved by showing that any kind of NG-group in the theorem be isomorphic to a permutation group on a quotient set of $A$ with respect to an equivalence relation on $A$.

Keywords: NG-group; Permutation group; Equivalence relation; $\chi$-subgroup

Introduction

This study considered the problem that the maximal order of a group consisting of transformations on a nonempty set $A$ and the group has no bijection as its element. Recall a permutation group on $A$ is a group consisting of bijections from $A$ to $A$ with respect to compositions of mappings. It is well known that any permutation group on a set $A$ with cardinality $n$ has an order not greater than $n!$.

In previous studies, there are some authors [1,2], problem 1.4 in [3] considering groups which consist of non-bijective transformations on $A$ where the binary operation is the composition of mappings. Our first result is on the orders of such groups.

Theorem 1.1. Let $A$ be a set with cardinality $n$. Suppose $NG$ be groups consisting of non-bijective transformations on $A$, where the binary operation on $NG$ is the composition of transformation. Then the order of $NG$ is not greater than $(n-I)!$ and there are such groups having order$(n-I)!$.

Then it was proven Theorem 1.1 by showing that any kind of group in the theorem be isomorphic to a permutation group on a quotient set of $A$ with respect to an equivalence relation on $A$. 
Definition 1.1. A class of group $\chi$ is called an SHP-class if it is closed under taking subgroups, homomorphic images, and products of normal subgroups. The latter condition means that if $U$ and $V$ are normal in $G$ and both $U$ and $V$ lie in $\chi$, then $UV \in \chi$. If a group $G$ belongs to $\chi$, we will say $G$ is an $\chi$-group.

Remark 1.2. If $\chi$ is an SHP-class and $U,V \lhd G$ are such that $G/U$ and $G/V$ are $\chi$-groups, then $G/(U\cap V)$ is isomorphic to a subgroup of the $\chi$-group $(U/G)\times(G/V)$, and thus $G/(U\cap V)$ is an $\chi$-group. It follows that given a finite group $G$, there exists a unique smallest normal subgroup $N$ such that $G/N \in \chi$, and we write $N = G\chi$. The following theorem was found by the author; see also lemma 2.32 in [4].

Theorem 1.2. Let $\chi$ be an SHP-class, and suppose $G = UV$, where $U$ and $V$ are subnormal in $G$. Then $G\chi = U\chi \times V\chi$. It could take the SHP-class to the class of $p$-groups, the class of solvable groups, etc. Theorem 1.2 will imply Lemma 9.15, problem 9B.5, Corollary 9.27, problem 9C.2, as corollaries.

Remark 1.3. It was noted in sec.4 of [5] that if it replaces the condition that $\chi$ is an SHP-class by some weaker condition that the class $\chi$ is such that whose composition factors all lie in some given set of simple groups then theorem 1.2 will fail in this case.

Definition 1.2. Let $\chi$ be an SHP-class and $G$ be a finite group. The result was denoted the maximal normal $\chi$-subgroup of $G$ by $G\chi$. Then was considered the question that if $G = UV$ with $U,V$ subnormal in $G$ then it holds that $G\chi = U\chi \times V\chi$ or not. If $p$ is a prime and take the SHP-class $\chi$ to be the class of all finite $p$-group, then for any finite group $G\chi$ will be $O_p(G)$ and results have the following theorem.

Theorem 1.3. Let $p$ and $q$ be two primes such that $q \equiv 1(\text{mod} \ p)$. Let $N = G\chi$ be a cyclic group of order $q$ and $H = \langle x^q \rangle \times \langle y^q \rangle$ an elementary abelian group of order $p^2$. Let $\langle x \rangle$ act on $N$ faithfully and $\langle y \rangle$ act on $N$ trivially. Set $G = N \rtimes H$ to the semidirect product of $N$ and $H$. Let $U = N \langle x \rangle$ and $V = \langle xy \rangle$. Then [1] $U,V$ are both subnormal in $G$. [2] $O_p(G) = \langle y \rangle$ and $O_p(U) = O_p(V) = 1$. In particular, $O_p(G) \neq O_p(U)O_p(V)$.

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Definition 2.1. A binary relation $\sim$ in $A$ is called an equivalence relation on $A$. If it satisfies the following three conditions:

(i) $a \sim a$ for any $a \in A$;
(ii) for any $a,b \in A$, if $a \sim b$ then $b \sim a$;
(iii) for any $a,b,c \in A$, if $a \sim b$ and $b \sim c$ then $a \sim c$.

For the set $A$, where use $A^A$ to denoted the set of all its transforms, for any $f \in A^A$, we use $\text{Im}(f)$ to denote the image of $f$. Also, $Z$ and $Z_e$ will respective denoted the set of integers and positive integers.

Definition 2.2. Let $\sim$ be an equivalence relation on $A$, for an element $a \in A$, it is call $\{x \in A | x \sim a\}$ the equivalence class of a determined by $\sim$, which is denoted by $[a]_\sim$. And $A/\sim = \{[a]_\sim | a \in A\}$ is called the quotient set of $S$ relative to the equivalence relation $\sim$.

Lemma 2.3. Theorem 1 [1], for any $f \in G$ and the $e$ the identity element of $G$, $\sim = \sim_f$.
Proof, for any \( a \in A \), so result goal is to show that \([a]_{e} = [a]_{e}\).

On one hand, if \( x \in [a]_{e} \), i.e. \( f(x) = f(a) \). Since \( G \) is a group with identity element \( e \), there is a transformation \( f' \in G \) such that \( f' = e = f \). Therefore,

\[
e(x) = f'(f(x)) = f'(f(a)) = e(a),
\]

Which yields that \( x \in [a]_{e} \).

On the other hand, if \( y \in [a]_{e} \), i.e. \( e(a)e(y) \). Hence,

\[
f(a) = (fe)(a) = ((fe)(y) = f(y),
\]

Which implies \( y \in [a]_{e} \). It follows that \([a]_{e} = [a]_{f}\) for any \( a \in A \), as wanted.

**Remark 2.1.** For Lemma 2.4, the current result see that \( \sim = \sim_{g} \) for any element \( f, g \in G \).

The following two corollaries are from [1], and we make some corrections to the original proofs. Actually, this adopt the restriction of finiteness on \( A \) in the first corollary from the original one. And then used the finiteness on \( A \) in the second corollary; the original one did not use it.

**Corollary 2.6.** Let \( f \) be an element in \( A^{A} \). Then \( \hat{f} = f \) iff the induced mapping \( \hat{f} \) on \( A/\sim_{f} \) is the identity element.

**Proof.** On one hand, suppose that \( \hat{f} = f \). Then for any \([x]_{f} \in A/\sim_{f} \), as \( f(x) = f([x]_{f}) \), then see that \([x]_{f} = [f(x)]_{f} \). It follows that

\[
\hat{f}([x]_{f}) = [f(x)]_{f} = [x]_{f};
\]

This implies that \( \hat{f} \) is the identity mapping on \( A/\sim_{f} \).

On the other hand, assume that \( \hat{f} \) is the identity mapping on \( A/\sim_{f} \). Then for any \([x]_{f} \in A/\sim_{f} \), the condition that \( \hat{f}([x]_{f}) = [x]_{f} \) will imply that \([f(x)]_{f} = [x]_{f} \) and hence \( f([x]_{f}) = f(x) \). It follows that \( \hat{f} = f \) as required.

**Corollary 2.7.** Suppose that \( A \) is a finite set and \( f \) is an element in \( A^{A} \). Then there is a group \( G \subseteq A^{A} \) containing \( f \) as an element iff \( \text{Im}(f) = \text{Im}(f') \).

**Proof.** On one hand, suppose that there is a group \( G \subseteq A^{A} \) containing \( f \) as an element. Let \( e \) be the identity element of \( G \). Then by Theorem 2.5, the induced mapping \( \hat{f} \) is a bijection on \( A/\sim_{f} \). In particular, \( \hat{f} \) is surjective and thus for any \( x \in A \), there is a \([y]_{f} \in A/\sim_{f} \) such that \( \hat{f}([y]_{f}) = [x]_{f} \) and hence \( f([y]_{f}) = f(x) \).

Which yields that \( f(x) = f([y]_{f}) = (f')([y]_{f}) \). As a result, \( \text{Im}(f) \subseteq \text{Im}(f') \) and thus \( \text{Im}(f) = \text{Im}(f') \).

On the other hand, suppose that \( \text{Im}(f) = \text{Im}(f') \). Thus, for any \( f(x) \in \text{Im}(f) \) there is a \( y \in A \) such that \( f(x) = f(y) \) and hence \( \hat{f}([y]_{f}) = [x]_{f} \); which implies that \( \hat{f} \) is surjective on \( A/\sim_{f} \). Note that results are assuming that \( A \) is finite and so is \( A/\sim_{f} \). This study has that the induced mapping \( \hat{f} \) is bijective. By Theorem 2.5, the assertion
follows.

**Remark 2.2.** Let $G \subseteq A^4$ be a group. That has seen, in Remark 2.1, that $\sim_F$ for any elements in $G$ and we will denote the common equivalence relation by $\sim$. Also, by Theorem 2.5, each element $f \in G$ will induce a bijection $\hat{f}$ on $A/\sim$.

The following theorem is crucial since it turns a group $G \subseteq A^4$ into a permutation group.

**Theorem 2.8.** Let $G \subseteq A^4$ be a group. Set $\hat{G} = \{ \hat{f} \mid f \in G \}$; then $\hat{G}$ is a permutation group on $A/\sim$ and $\rho : G \rightarrow \hat{G}, f \mapsto \hat{f}$, is an isomorphism. *Proof.* For any $f, g \in G$ and any $[a] \in A/\sim$, results have $\rho(fg)([a]) = ([fg](a)) = [f(g(a))] = \rho(f)\rho(g)([a])$; which implies that $\rho(fg) = \rho(f)\rho(g)$ and thus $\rho$ is a homomorphism.

By the definition of $\hat{G}$, it is obvious that $\rho$ is surjective.

Now suppose that $\rho(f) = \rho(g)$ for two elements $f, g \in G$, i.e. $[f(a)] = [g(a)]$, $\forall a \in A$: Let $e$ be the identity element of $G$, then we have $[f(a)]e = [g(a)]e$; $\forall a \in A$. It follows that $e(f(a)) = e(g(a)); \forall a \in A$. Hence, $f(a) = (ef)(a) = e(f(a)) = e(g(a)) = g(a), \forall a \in A$, and therefore $f = g$, so it conclude that $\rho$ is injective. As a consequence, $\rho$ is an isomorphism.

**Definition 2.3.** A subgroup $H$ of a group $G$ is called characteristic in $G$, denoted $H \text{ char } G$, if every automorphism of $G$ maps $H$ to itself, that is $\rho(H) = H$ for all $\rho \in \text{Aut}(G)$.

**Remark 2.3.** If $H$ is characteristic in $G$ in $K$ and $K$ is characteristic in $G$, then $H$ is characteristic in $G$.

Let $G$ be a finite group. It has the following two lemmas. They are from Section 2 of (5).

**Lemma 2.9.** Suppose that $\chi$ is an SHP-class.

(a) Let $\leq G$ be a subgroup. Then $H^\chi \leq G^\chi$.

(b) Let $N < G$ be a normal subgroup of $G$ and write $\bar{G} = G/N$. then $\bar{G}^\chi = \bar{G}^\chi$.

(c) $G^\chi$ is characteristic in $G$.

(d) $O^\chi(G)$ is characteristic in $G$.

The following lemma is a generalization of Problem 2A.1 in (8).

**Lemma 2.10.** Let $A$ and $B$ be two subnormal $\chi$-subgroups of $G$. Then the subgroup $<A, B>$ generated by $A$ and $B$ are $\chi$-subgroup of $G$.

*Proof.* Let $A$ be a subnormal $\chi$-subgroup of $G$. The resulting use induction on the subnormal depth $r$, $A \subseteq O^\chi_r(A)$ in $G$ to show that if $r = 1$, then $A$ and $A \subseteq O^\chi_r(A)$ since $O^\chi_r(G)$ is the largest normal $\chi$-subgroup of $G$.

Suppose $r > 1$ and the containment holds for $r-1$. Let $A_1 = A \lhd \ldots \lhd H_{r-1} \lhd H_r = G$ be a subnormal series from $A$ to $G$: Then $A \subseteq O^\chi_r(G)$ by inductive hypothesis. Since $O^\chi_r(G)$ acts on $H_{r-1}$ and $H_{r-1} \lhd G$; $O^\chi_r(G) \lhd G$ and then $O^\chi_r(G)/(H_{r-1}) \subseteq O^\chi_r(G)$.

It was concluded that $A \subseteq O^\chi_r(G)$.

In general, for any two subnormal $\chi$-subgroups $A$ and $B$, $A, B \subseteq O^\chi_r(G)$ and thus $<A, B> \subseteq O^\chi_r(G)$ as wanted.
Proofs of Main Results

Now let $A$ be a set having $n$ letters written as $\{1, 2, \ldots, n\}$. The results have the following theorem, which is Theorem 1.1.

**Theorem 3.1.** Let $A$ be a set with cardinality $n$ with $n \geq 3$. Suppose $NG$ is a group consisting of non-bijective transformations on $A$, where the binary operation on $NG$ is the composition of transformations. Then the order of $NG$ is not greater than $(n-1)!$ and there are such groups having order $(n-1)!$.

**Proof.** Let $NG$ be a group consisting of non-bijective transformations on $A$. By Remark 2.1, it is known that $\approx = \approx_g$ for any element $f, g \in NG$ and it denote the common equivalence relation by $\sim$. Note that $NG$ is a group consisting of non-bijective transformations, then we see that the equivalence relation is not the equality relation $= \equiv$ $A$. Thus, these results have that the quotient set $A/\sim$ has an order less than $n-1$.

Additionally, $NG$ is isomorphic to a permutation group on $A/\sim$ by Theorem 2.8. It follows that the order of $NG$ is less than $(n-1)!$ as any permutation group on $A/\sim$ has order less than $(n-1)!$.

Note that In defining a permutation $s$ on the set $\{1, 3, \ldots, n\}$, there are $n-1$ choices for $\rho(1)$, $n-2$ choices of $\rho(3) \neq \rho(1), n-2$ choices of $\rho(4) (\neq \rho(1), \rho(3))$, etc., i.e. totally $(n-1)(n-2)! = (n-1)!$.

**Theorem 3.2.** Let $\chi$ be an SHP-class, and suppose $G = UV$, where $U$ and $V$ are subnormal in $G$. Then $G^\chi = U^\chi V^\chi$.

**Proof.** This work use induction of the subnormal depth of $U$ in $G$ to prove the result.

First, if the subnormal depth of $U$ in $G$ is one, i.e. $U \triangleleft G$. Since $U^\chi$ is characteristic in $U$ and $U$ is normal in $G$ we see that $U^\chi$ is normal in $G$.

Let $G/\chi / U^\chi$. By the hypothesis, $\bar{G} = \bar{U} \cap \bar{V}$ where $\bar{U} = U/ U^\chi$, $\bar{V} = V/ U^\chi$.

Thus, $\bar{U}$ is a normal $\chi$-group of $\bar{G}$ and $\bar{V}$ is subnormal in $G$. By Lemma 2.10, we have $\bar{G}^x = \bar{V}^x$. By Lemma 2.9 (b), $\bar{G} = G^\chi$, $\bar{V}^x = \bar{V} = \bar{U}^x \bar{V}^x$. By correspondence theorem, it has $G^\chi = U^\chi V^\chi$; as required.

Now suppose that the subnormal depth of $U$ in $G$ is $r$ with $r > 1$: Let $U_1 = U \triangleright U_2 \triangleright \ldots \triangleright U_r \triangleright G$

be a subnormal series from $U$ to $G$ with length $r$. By Dedekind's lemma,

$U_i = U(V \cap U_i)$. As both $U$ and $V$ are subnormal in $U$; and $U$ has subnormal depth $r-1$ in $U_i$, then obtain that

$(U_i)^\chi = U^\chi (V \cap U_i)^\chi$

by inductive hypothesis. Also, $G = U,V$ with $U$ normal in $G$ and $V$ subnormal in $G$, and hence

$G^\chi = (U_i)^\chi V^\chi = U^\chi (V \cap U_i)^\chi V^\chi = U^\chi V^\chi$,

because $(V \cap U_i)^\chi \subseteq V^\chi$ by Lemma 2.9 (a).

**Theorem 3.3.** Let $p$ and $q$ be two primes such that $q = 1 \pmod{p}$. Let $N = C_q$

be a cyclic group of order $q$ and $H = \langle \rangle$ an elementary abelian group of order

$p^2$. Let $\langle \rangle$ act on $N$ faithfully and $\langle \rangle$ act on $N$ trivially. Set $G = N \rtimes H$ to be the semidirect product of $N$ and $H$. Let $U = N \langle \rangle$ and $V = N \langle \rangle$. Then

(i) $U, V$ are both subnormal in $G$ and $G = UV$. 


(ii) $O_p(G) = \langle y \rangle$ and $O_p(U) = O_p(V) = 1$. In particular, $O_p(G) \neq O_p(U)O_p(V)$.

**Proof.** Since $N$ is normal in $G$ and the quotient group $G=N/H$ is abelian, it deduced that the derived subgroup $G'$ is contained in $N$. It follows that both $U$ and $V$ contain $G'$ as a subgroup, which implies that $U$ and $V$ are normal in $G$. Obviously, $G = UV$. Assertion (i) holds.

Note that the Sylow $p$-subgroup of $G$ is not normal since $\langle x \rangle$ act on $N$ faithfully and hence $O_p(G)$ has an order less than $p^2$. However, as $\langle y \rangle$ act on $N$ trivially, $N$ normalizes $\langle y \rangle$ which yields that $\langle y \rangle$ is a normal $p$-subgroup of $G$. It is easy to see that $O_p(G) = \langle y \rangle$. Both $\langle x \rangle$ and $\langle xy \rangle$ act faithfully on $N$, which yields that $O_p(U) = O_p(V) = 1$; as wanted.

**References**